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The Method of Iteration

To find the real roots of the eqn $f(x) = 0$ — ①
by iteration Method we first express the given equation in the form $x = \phi(x)$ — ②

First we take an approximate value of the desired root suppose x_0, we substitute this value in $\phi(x)$ and get a better approximation x_1 i.e., $x_1 = \phi(x_0)$ — ③

Putting $x = x_1$ in $\phi(x)$ and get the next better approximation x_2 by the eqn $x_2 = \phi(x_1)$ — ④

Similarly proceeding we get successive approximations i.e.,

$$x_3 = \phi(x_2)$$
$$x_4 = \phi(x_3)$$
$$\cdots\cdots$$
$$x_n = \phi(x_{n-1}), n = 1, 2, 3, \ldots \quad ⑤$$

Note:-

From $f(x) = 0$, we may get the form $x = \phi(x)$ in many ways, for example the eqn $x^3 + x^2 - 1 = 0$ can be expressed as

(i) $x = (1+x)^{-1/2}$

(ii) $x = (1-x^3)^{1/2}$

(iii) $x = (1-x^2)^{1/3}$

Also the iterative method is applicable only if $|\phi'(x)| < 1$ for $x \in (a, b)$

Ques Find a real root of the equation by Iteration method. $f(x) = x^3 + x^2 - 1 = 0$

Soln Obviously $f(0) = -1 < 0$

$f(1) = 1 > 0$

$\because$ the root of $f(x) = 0$ lies b/w 1 and 0

Also the given eqn can be rewritten as

$$x = \frac{1}{\sqrt{x+1}} \quad \text{——— (1)}$$

$$\Rightarrow \phi(x) = \frac{1}{\sqrt{x+1}} \Rightarrow \phi'(x) = \frac{-1}{2(x+1)^{3/2}}$$

$$\Rightarrow |\phi'(x)| < 1 \quad \text{as} \quad 0 < x < 1$$

Thus the iterative method is applicable

$\therefore$ we have $x_{n+1} = \phi(x_n) \; \forall n = 0, 1, 2, 3, \ldots$

then $x_0 = \frac{0+1}{2} = 0.5$

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1+0.5}} = 0.8165$$

and $x_2 = \phi(x_1) = \frac{1}{\sqrt{1+0.8165}} = 0.7419$

and $x_3 = \phi(x_2) = \frac{1}{\sqrt{1+0.7419}} = 0.7577$

and $x_4 = \phi(x_3) = \frac{1}{\sqrt{1+0.7577}} = 0.7543$

and $x_5 = \phi(x_4) = \frac{1}{\sqrt{1+0.7543}} = 0.7550$

and $x_6 = \phi(x_5) = \frac{1}{\sqrt{1+0.755}} = 0.7548$

and $x_7 = \phi(x_6) = \frac{1}{\sqrt{1+0.7548}} = 0.7548$.

$\because x_6 = x_7$. Hence the required real root is 0.7548

Ans

Ques Find the root of the eqn. $2x = \cos x + 3$ correct upto three decimal places by Iteration method.

Soln Let $f(x) = 2x - \cos x - 3 = 0$

so that $f(0) = -4 < 0$

$f(1) = -1.54 < 0$

$f(2) = 1.416 > 0$

$\therefore$ The real root of $f(x)=0$ lies b/w 1 and 2

$\Rightarrow x_0 = \frac{1+2}{2} = 1.5$

Also $f(x)=0$ can be rewritten as

$x = \frac{\cos x + 3}{2} \Rightarrow \phi(x) = \frac{\cos x + 3}{2}$

$\Rightarrow \phi'(x) = -\frac{1}{2}\sin x$

$\Rightarrow |\phi'(x)| = \left|\frac{\sin x}{2}\right| < 1 \quad \forall x \in (1,2)$

$\therefore$ The iteration method is applicable

$\Rightarrow x_1 = \phi(x_0)$

$= \frac{\cos x_0 + 3}{2} = \frac{\cos(1.5)+3}{2} = 1.5354$

and $x_2 = \phi(x_1)$

$= \frac{\cos(1.5354)+3}{2} = 1.5177$

and $x_3 = \phi(x_2)$

$= \frac{\cos(1.5177)+3}{2} = 1.5265$

and $x_4 = \phi(x_3)$

$= \frac{\cos(1.5265)+3}{2} = 1.5221$

and $x_5 = \phi(x_4)$

$$= \frac{\cos(1.5221)+3}{2} = 1.5243$$

and $x_6 = \phi(x_5)$

$$= \frac{\cos(1.5243)+3}{2} = 1.5232$$

and $x_7 = \phi(x_6)$

$$= \frac{\cos(1.5232)+3}{2} = 1.5237$$

Hence, the root of the equation $f(x)=0$ upto three decimal places is 1.523

Ans

Ques If α, β are the roots of $x^2+ax+b=0$, Show that $x_{n+1} = -\left(\frac{ax_n+b}{x_n}\right)$

will converge near $x=\alpha$, if $|\alpha| > |\beta|$ and the iteration $x_{n+1} = \frac{-b}{x_n+a}$ will converge near $x=\alpha$, if $|\alpha|<|\beta|$.

Soln If $x_{n+1} = \phi(x_n)$, then by iteration method this form will converge if $|\phi'(x)| < 1$ at $x = x_n$.
Since α, β are the roots of
$x^2+ax+b=0$ so that $\alpha+\beta = -a$, $\alpha\beta = b$.

(i) Now, for the iteration $x_{n+1} = -\left(\frac{ax_n+b}{x_n}\right)$ — ①
We have $\phi(x) = -\frac{(ax+b)}{x}$

$\therefore$ Eqn ① will converge near $x=\alpha$ if

$$\left[\left|\frac{d}{dx}\left[-\frac{(ax+b)}{x}\right]\right|\right]_{x=x_n} < 1$$

$$\Rightarrow \left|\left(\frac{b}{x_2}\right)_{x=x_n}\right| < 1 \quad \Rightarrow \left|\frac{b}{x_n^2}\right| < 1$$

$$\Rightarrow |b| < |x_n^2| \Rightarrow x_n^2 > |b|$$

$$\Rightarrow |\alpha|^2 > |b| \quad \text{———} \quad (\because x_n \to \alpha)$$

$$\Rightarrow |\alpha|^2 > |\alpha\beta| \quad \text{———} \quad (\because \alpha\beta = b)$$

$$\Rightarrow |\alpha|^2 > |\alpha| \cdot |\beta|$$

$$\Rightarrow |\alpha| > |\beta|$$

(ii) For the iteration

$$x_{n+1} = \frac{-b}{x_n + a} \quad \text{——} (2)$$

we have $\phi(x) = \dfrac{-b}{x+a}$

$\therefore$ Eqn (2) will converge near $x = \alpha$ if

$$\left|\frac{d}{dx}\left(\frac{-b}{x+a}\right)\right| < 1 \quad \text{at } x = x_n.$$

$$\Rightarrow \left|\frac{b}{(x_n+a)^2}\right| < 1 \Rightarrow (x_n+a)^2 > |b|$$

$$\Rightarrow (\alpha + a)^2 > |b| \quad \text{———} \quad (\because x_n \to \alpha)$$

$$\Rightarrow \beta^2 > |b| \quad \text{———} \quad (\because \alpha + \beta = -a)$$

$$\Rightarrow |\beta|^2 > |\alpha\beta| \quad \text{———} \quad (\because \alpha\beta = b)$$

$$\Rightarrow |\beta|^2 > |\alpha||\beta|$$

$$\Rightarrow |\beta| > |\alpha|$$

Proved

3.3.3. Iteration Method For the System of Non-Linear Equations :

Let
$$\left.\begin{aligned} f(x, y) = 0 \\ g(x, y) = 0 \end{aligned}\right\} \quad \ldots(1)$$

be a system of non-linear equations whose real roots are to be required within the degree of accuracy.

From (1) we may take
$$x = F(x, y) \text{ and } y = G(x, y) \quad \ldots(2)$$

provided
$$\left|\frac{\partial F}{\partial x}\right| + \left|\frac{\partial F}{\partial y}\right| < 1 \text{ and } \left|\frac{\partial G}{\partial x}\right| + \left|\frac{\partial G}{\partial y}\right| < 1$$

in the neighbourhood of the root.

Let (ξ, η) be the exact root of (1) and let (x_0, y_0) be its initial approximation. Then from (2) we obtain successive approximations as,
$$x_1 = F(x_0, y_0); \quad y_1 = G(x_0, y_0)$$
$$x_2 = F(x_1, y_1); \quad y_2 = G(x_1, y_1)$$
$$x_3 = F(x_2, y_2); \quad y_3 = G(x_2, y_2).$$

Continuing this process, we get
$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = G(x_n, y_n)$$

as two iterative formulae. If those formulae converge, then in the limit
$$\xi = F(\xi, \eta) \text{ and } \eta = G(\xi, \eta)$$

thus, (ξ, η) gives the root of system (1).

Example 8. *Find a real root of the system of the equations by iteration method,*
$$x = 0{\cdot}2x^2 + 0{\cdot}8, \quad y = 0{\cdot}3xy^2 + 0{\cdot}7.$$

Solution. Assume that
$$\left.\begin{aligned} F(x, y) = 0{\cdot}2x^2 + 0{\cdot}8 \\ \text{and} \quad G(x, y) = 0{\cdot}3xy^2 + 0{\cdot}7 \end{aligned}\right\}. \quad \ldots(1)$$

$$\therefore \quad \frac{\partial F}{\partial x} = 0{\cdot}4x, \; \frac{\partial F}{\partial y} = 0, \; \frac{\partial G}{\partial x} = 0{\cdot}3y^2, \; \frac{\partial G}{\partial y} = 0{\cdot}6xy.$$

Choosing that $x_0 = \frac{1}{2}, y_0 = \frac{1}{2}$, then
$$\left|\frac{\partial F}{\partial x}\right|_{(x_0, y_0)} + \left|\frac{\partial F}{\partial y}\right|_{(x_0, y_0)} = 0{\cdot}2 < 1$$

and
$$\left|\frac{\partial G}{\partial x}\right|_{(x_0, y_0)} + \left|\frac{\partial G}{\partial y}\right|_{(x_0, y_0)} = \frac{0{\cdot}3}{4} + \frac{0{\cdot}6}{4} = \frac{0{\cdot}9}{4} < 1.$$

Thus, for (x_0, y_0) the condition are satisfied. Therefore, we further obtain the successive approximations as

Ist approximation. $x_1 = F(x_0, y_0) = 0{\cdot}2\left(\frac{1}{2}\right)^2 + 0{\cdot}8 = 0{\cdot}85$

$$y_1 = G(x_0, y_0) = 0{\cdot}3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + \ldots$$

IInd approximation. $x_2 = F(x_1, y_1) = 0.2\, x_1^2 + 0.8$

$$= 0.2\,(0.85)^2 + 0.8 = 0.9445$$

$$y_2 = G(x_1, y_1) = 0.3\, x_1 y_1^2 + 0.7$$

$$= 0.3\,(0.85)\,(0.74)^2 + 0.7 = 0.8396.$$

IIIrd approximation. $x_3 = F(x_2, y_2) = 0.2\, x_2^2 + 0.8$

$$= 0.2\,(0.9445)^2 + 0.8 = 0.9784$$

$$y_3 = G(x_2, y_2) = 0.3\, x_2 y_2^2 + 0.7$$

$$= 0.3\,(0.9445)\,(0.8396)^2 + 0.7 = 0.8997.$$

From three approximations we conclude that the root converges to (1, 1). Also from (1), we obtain

$$1 = F(1, 1) \text{ and } 1 = G(1, 1).$$

EXERCISE 2

1. Use the iteration method to find a real root of the following equations correct to four decimal places :

(i) $\cos x = 3x - 1$ (ii) $x = \dfrac{1}{(1+x)^2}$ (iii) $x = (5 - x)^{1/3}$

(iv) $\sin x = 10\,(x - 1)$ (v) $e^{-x} = 10x$ (vi) $x = \text{cosec}\, x$

(vii) $\sin^2 x = x^2 - 1$ (viii) $e^x = \cot x$ (ix) $1 + x^2 = x^3$

(x) $2x - \log_{10} x = 7$ (xi) $5x^3 - 20x + 3 = 0$ (xii) $\sin x = \dfrac{x+1}{x-1}$.

2. Use the method of iteration to find a positive root, lies between 0 and 1 of the equation $xe^x = 1$.

3. Find the smallest root of the equation,

$$1 - x + \frac{x^2}{(2\,!)^2} - \frac{x^3}{(3\,!)^2} + \frac{x^4}{(4\,!)^2} - \frac{x^5}{(5\,!)^2} + \ldots = 0.$$

4. If $F(x)$ is sufficiently differentiable and the iteration $x_{n+1} = F(x_n)$ converges, prove that the order of convergence is a positive integer.

5. Find the reciprocal of 41 correct to four decimal places by iteration formula

$$x_{n+1} = x_n\,(2 - 41x_n).$$

6. Find the cube root of 15 correct to four significant figures by iteration method.

7. Find by iteration method, the real root of the equation $3x - \log_{10} x = 6$ correct to four significant figures.

8. Solve by iteration method :

(i) $1 + \log x = \dfrac{x}{2}$ (ii) $x^3 + x + 1 = 0$

(iii) $x^3 = x^2 + x + 1$, near $x = 2$ (iv) $x^3 - 2x^2 - 5 = 0$

(v) $x^3 - 2x^2 - 4 = 0$.

9. By iteration method, find the square root of 30.

ANSWERS

1. (i) 0·6071, (ii) 0·4656, (iii) 1·5159, (iv) 1·0886,
(v) 0·09128, (vi) 1·1142, (vii) 1·4044, (viii) 0·5314,
(ix) 1·4331, (x) 3·7893, (xi) 0·1508, 1·920 (xii) – 0·42037

2. $x_{16} = x_{17} = 0{\cdot}5672$ **3.** 1·44 **5.** 0·0244 **7.** 2·1080

8. (i) 2·935305 (ii) 0·341163 (iii) 1·83928 (iv) 2·690647 (v) 2·594313

9. 5·477225.